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On the deformation of periodic long waves over a gently sloping bottom

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Two-dimensional time-periodic water waves on a gently sloping bottom are investigated under the classical long-wave assumptions that $\epsilon = h_0'/\lambda'$ and $\delta = H'/h_0'$ are small parameters (H' being the wave height, h_0' a characteristic water depth and λ' the horizontal scale for the oscillatory motion) and the assumption that $\delta/\epsilon^2 = O(1)$ as δ and ϵ tend to zero.

It is shown that for a bottom slope h_x for which $h_x = o(\epsilon^3)$ the governing KdV equation with slowly varying coefficients (derived by Johnson 1973*a*) has a time-periodic solution which in the first approximation is a slowly varying cnoidal wave. The second-order approximation in an asymptotic expansion with respect to the bottom slope represents the deformation of this wave due to the sloping bottom. For $h_x \gg \epsilon^5$ this deformation is larger than the second-order contribution from the basic expansion with respect to wave amplitude which underlies the KdV equation itself; the calculation is then a consistent approximation to physical reality.

Numerical results for the deformation are given. Also, the wave profiles are compared with experiments on a plane of slope $h_x = \frac{1}{35}$ and show good agreement even for the large values of H'/h' appropriate to waves rather close to breaking.

1. Introduction

The literature on long waves over a sloping bottom has grown steadily during the last decade, beginning with the numerical solution of the equations by Peregrine (1967), and continuing with the numerical work of Madsen & Mei (1969*a*, *b*), who based their analysis on the equations derived by Mei & LeMehauté (1966). A strictly numerical approach has been used by Street, Burges & Whitford (1968).

The work presented here is concerned with a part of the problem which has so far been little investigated, with the exception of studies by Ostrovskiy & Pelinovskiy (1970), Svendsen & Brink-Kjær (1972) and Svendsen & Buhr Hansen (1977), namely the slow transformation and deformation of a periodic wave train over a gently sloping bottom. (Grimshaw (1970) and Johnson (1973*a*, *b*) considered the same problem for a solitary wave.)

Madsen & Mei assumed that the bottom slope h_x was of the same order of magnitude as the parameter $\epsilon = h_0'/\lambda'$, λ' being a characteristic horizontal length of the wave motion (the wavelength or 'length of a wave crest', say) and h_0' a characteristic water depth (primes denoting dimensional variables). This implies that appreciable changes in depth occur over the distance λ', and the bottom must therefore be termed steeply sloping.

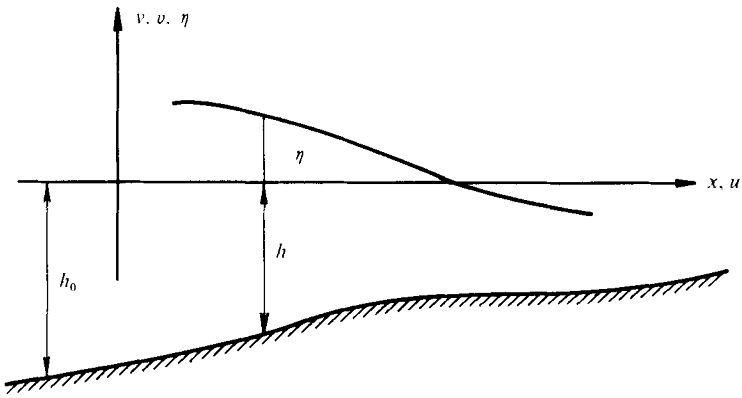

FIGURE 1. Definition sketch.

Their computations showed that a solitary wave propagating over a linear transition from one constant water depth to another, much smaller constant water depth would breed smaller trailing waves, which owing to their smaller height would propagate at a lower speed. A similar result was obtained for the individual wave crests in a periodic wave train. A physical explanation seems to be that, on passing over a slope which is too steep, the wave is unable to adjust its initially stable form to the rapidly changing local depth. Thus the disintegration of the wave when it is propagating on the shelf is analogous to that found by Zabusky & Kruskal (1965) on a horizontal bottom using an initial wave shape which is not a constant-form solution to the horizontal-bottom KdV equation.

This will happen even for waves of rather small amplitude-to-depth ratio ($H'/h' = 0{\cdot}12$ in the case described by Madsen & Mei). In nature, however, waves on a sloping breach are often seen to develop gradually, their heights growing to the order of the water depth without any (visible) sign of disintegration, before they eventually break. Obviously this entirely different behaviour is a consequence of a more gentle bottom slope.

This is clearly seen by looking at the mathematical description. If $h_x = o(\epsilon)$ the equations used by Peregrine and by Madsen & Mei (which are Boussinesq-type equations assuming $\eta'\lambda'^2/h_0'^3 = O(1)$) can be simplified considerably and actually reduced to a KdV-type equation in the surface elevation η (Johnson 1973*a*). In terms of variables (figure 1) that have been non-dimensionalized by the length λ' and the time $\lambda'/(gh_0')^{\frac{1}{2}}$ (g being the acceleration due to gravity) this equation reads

$$\eta_t + c\eta_x + \tfrac{3}{2}ch^{-1}\eta\eta_x + \tfrac{1}{6}h^2c\eta_{xxx} + \tfrac{1}{2}c_x\eta = 0, \qquad (1)$$

where $c = (h/\epsilon)^{\frac{1}{2}}$, h being dimensionless and therefore $O(\epsilon)$. The last term in (1) represents the influence of the depth variation, the derivative c_x being equal to $h_x c/2h$; c and h are (slowly varying) functions of x.

On a horizontal bottom this equation reduces to the well-known KdV equation with constant coefficients, with cnoidal waves as periodic solutions of constant form. The nonlinear term and the η_{xxx} term (representing amplitude and frequency dispersion respectively) will in this case be comparable, which implies that they are $O(\epsilon^5)$ (in the chosen non-dimensional system) and that η, η_t and $c\eta_x$ are $O(\epsilon^3)$. Applying these results to (1) shows that the last term will be $O(h_x\epsilon^2)$.

If we now let $h_x \gtrsim \epsilon^3$ (but still $o(\epsilon)$) the last term is $\gtrsim \epsilon^5$ and hence can destroy the balance between amplitude and frequency dispersion. Therefore $\epsilon^3 \lesssim h_x \leqslant \epsilon$ implies a radical distortion of the wave profile due to the sloping bottom as was also shown by Johnson (1972, 1973*a*), who investigated a solitary wave in this case and found that the waves disintegrate into solitons in much the same way as was found by Madsen & Mei for the even steeper slope $h_x = O(\epsilon)$.

The last term in (1) is only indirectly responsible for the disintegration of the waves. As shown by Zabusky & Kruskal (1965), even the horizontal-bottom KdV equation has the property that an initial surface distribution is unstable if it is not a constant-form solution to that equation. Thus if the c_x term in (1) is $\gtrsim \epsilon^5$, its role will essentially be to disturb the initially stable profile and hence produce conditions which even on a horizontal bottom would make the wave disintegrate into solitons.

On the other hand, a slow and (up to a certain point) stable development of the wave profile, such as precedes breaking on a gentle beach, obviously requires that the last term in (1) causes only a minor disturbance of the wave profile. In other words, what is felt by the wave as a gently sloping bottom requires that $\epsilon^2 h_x = o(\epsilon^5)$ as $\epsilon \to 0$, or $h_x = o(\epsilon^3)$, and this is the case we shall consider here. Then an asymptotic expansion can be constructed (§ 2) in which the first approximation is a cnoidal solution of the local horizontal-bottom equation. This is equivalent to the classical shoaling solution first considered for linear waves on an intuitive physical basis by Rayleigh (1911). For cnoidal waves the analogous problem was discussed by Ostrovskiy & Pelinovskiy (1970) and later by Svendsen & Brink-Kjær (1972).

The second approximation in the asymptotic expansion (which has not previously been obtained, to our knowledge) will represent the deformation of the wave due to the sloping bottom. We shall find in § 3 that this skews the wave such that the front side is steeper than the rear side when $h_x < 0$. In this approach the unperturbed (first-order) wave profile is determined only by the local water depth (and the initial data) while the deformation is a perturbation depending only on the first-order solution and is proportional to the local value of the bottom slope. Therefore, according to this solution, the deformation continuously adjusts itself to the local conditions as the wave propagates and remains small as long as h_x and H/h are sufficiently small. This also implies that the disturbance (within this approximation) is 'reversible', so that if the wave propagates onto a shelf with a horizontal bottom the surface profile will 'immediately' regain its symmetrical shape, in contrast to the situation demonstrated by Madsen & Mei (1969*b*) for a steeper slope. In their case the major part of the deformation actually takes place after the wave has reached the horizontal shelf.

On the other hand this behaviour clearly indicates that the theory cannot predict (even qualitatively) the last part of the process leading to breaking, where the actual deformation has an increasingly irreversible character involving memory of its past history. From a certain point the wave will break irrespective of the local slope.

Johnson (1973*b*) analysed the behaviour of a solitary wave on a slope by letting $h_x \to 0$ in the KdV equation (1).† It should be remembered, however, that the KdV

† Grimshaw (1970) also investigated the development of a solitary wave on a slope. However, he assumed a slowly varying solution without analysing for which magnitudes of h_x such a solution exists. In addition, after a certain step in the development he cancelled (as pointed out by Johnson 1973*a*) the ordering of the terms in the equations by simply putting the small parameters equal to unity. For these two reasons his final equations included various inconsistent small terms.

equation represents only the first approximation to the physical phenomenon in an asymptotic description with respect to the amplitude parameter $H/h = O(\epsilon^2)$. The complete asymptotic expansion for the physical problem is of the type

$$\eta = \sum_{j=0}^{M} \epsilon^{2j} \sum_{i=0}^{N} \sigma^i \eta^{(ij)},$$

where σ is a small parameter such that

$$\sigma = O(h_x \epsilon^{-3}) \tag{2}$$

and $\eta^{(0j)}$ corresponds to the general horizontal-bottom solution. The solution we obtain from the KdV equation (1) includes only $\eta^{(00)}$ and $\eta^{(10)}$. As indicated above, $\eta^{(00)} = O(\epsilon^3)$ and consequently $\epsilon^2\eta^{(01)}$ is $O(\epsilon^5)$. Hence, if we want consistently to neglect this second-order solution in the amplitude, we must require

$$\sigma\eta^{(10)} \gg \epsilon^2\eta^{(01)} = O(\epsilon^5).$$

If (as will be verified) $\eta^{(10)} \sim \epsilon^3$, this is equivalent to $\sigma \gg \epsilon^2$ and hence to $h_x \gg \epsilon^5$. Thus the description developed in Johnson (1973*b*) and Grimshaw (1970) and extended in the following is consistent only for

$$\epsilon^2 \ll \sigma \ll 1, \quad \text{i.e.} \quad \epsilon^5 \ll h_x \ll \epsilon^3. \tag{3}$$

Thus if we let $h_x \to 0$ we must require $\epsilon \to 0$ (and hence $H/h \to 0$) simultaneously to maintain consistency. In fact consistency could also be maintained for h_x fixed but smaller than ϵ^5 by adding the pertinent higher-order solutions $\eta^{(0j)}$ for the constant-depth case.

Johnson (1973*b*) found that, at the trailing end of a solitary-wave profile, oscillations develop whose amplitudes decay with increasing distance from the crest. The periodic solution studied here appears to be entirely different in that the periodicity conditions imposed on the deformed wave yield a continuous and smooth surface profile. At the same time the periodicity conditions supply the missing condition for the determination of the variable coefficients of the first-order solution, which is shown to correspond to the conservation of energy, i.e. to Rayleigh's assumption of constant energy flux.

The variation of the mean water level ('wave set-down') does not influence the solution to the order of magnitude considered, but may readily be derived from the equations, as may the radiation stress.

Numerical results will be given for $\sigma\eta^{(10)}$ (denoted below by $\sigma\eta^{(1)}$) and the theoretical wave profiles will be compared with experimental results. This is of particular interest as one of the questions left open by the theoretical treatment is the numerical range of parameter values for which the assumptions for h_x work in practice. It is shown that, for a plane of slope $h_x = \frac{1}{35}$ and values of λ'/h' which are not too large, the theoretical wave profiles agree remarkably well with the measured profiles, even for wave heights close to breaking. This also supports the general idea of using a cnoidal wave as a basis for the perturbation solution.

2. Asymptotic solution

The fluid is assumed to be incompressible and the flow to be two-dimensional and irrotational. Introducing co-ordinates and definitions as shown in figure 1, the non-dimensional system of variables introduced previously is used in the following. Note that h is assumed to be $O(\epsilon)$ and no smaller, i.e. we exclude conditions where $h \to 0$.

We then seek solutions to (1) in the range of h_x described by (3) which are uniformly valid for large ranges of x and t. To obtain these a two-scale method is chosen (e.g. Cole 1968, chap. 3; Mahony 1972) in which a fast variable θ and a slow variable X are defined by

$$\theta = \theta_t t + \int_0^x \theta_x dx, \quad X = \sigma\epsilon^2 x. \tag{4}$$

Here θ is a phase angle and θ_x is chosen as L^{-1}, where $L = O(1)$ is the local dimensionless wavelength; θ_t will be assumed constant (time-periodic waves). Thus the ratio $-\theta_t/\theta_x$ is the local phase velocity $[= c(1+O(\epsilon^2))]$.

The co-ordinate scaling (4) has been so arranged that $h_X \sim \epsilon \sim h$, as can be seen from (2) and (4).

We now regard η as a function of θ and X, with $\eta_X \sim \eta \sim \epsilon^3$, and take θ_x to be a function of X only. Posing the expansions

$$\eta = \eta^{(0)} + \sigma\eta^{(1)} + \dots, \tag{5a}$$

$$\theta_x = \theta_x^{(0)} + \sigma\theta_x^{(1)} + \dots \tag{5b}$$

yields upon substitution into (1) the equation for the leading approximation:

$$(\theta_t + c\theta_x^{(0)} + \tfrac{3}{2}ch^{-1}\theta_x^{(0)}\eta^{(0)})\,\eta_\theta^{(0)} + \tfrac{1}{6}h^2c\theta_x^{(0)3}\eta_{\theta\theta\theta}^{(0)} = 0. \tag{6}$$

This is the KdV equation for a horizontal bottom, and the solution appropriate to this case is a periodic (cnoidal) wave with

$$\eta^{(0)} = H\{\beta(m) + \mathrm{cn}^2(2K\theta, m)\} + O(\epsilon^5) \tag{7a}$$

and

$$\theta_x = -\theta_t/c_1, \tag{7b}$$

where

$$c_1 = c(1 + A(m)H/h)^{\frac{1}{2}}, \tag{7c}$$

$$\beta(m) = (1 - E/K)m^{-1} - 1, \tag{7d}$$

$$A(m) = (2 - 3E/K)m^{-1} - 1, \tag{7e}$$

$$U = HL^2/h^3 = \tfrac{16}{3}mK^2. \tag{7f}$$

$H = O(\epsilon^3)$ is the dimensionless wave height, $K = K(m)$ and $E = E(m)$ $(0 \leqslant m < 1)$ are the complete elliptic integrals of the first and second kind respectively, m is their parameter and cn is the Jacobian elliptic function. m (and hence K and E) is determined from the wave parameters in (7f). β represents the position of the mean water level relative to the wave trough and A is the amplitude dispersion. (The solitary wave is the special case of (7) corresponding to $m \to 1, K \to \infty, E \to 1$.)

This solution has two parameters, H and m, which in this context are functions of X. Their determination was first discussed by Ostrovskiy & Pelinovskiy (1970). Essentially $H(X)$ and $m(X)$ are determined by specification of the wave period (which we have assumed constant) and specification of H for some value of X, together with the conservation of energy. Within our expansion procedure the condition corresponding to the conservation of energy will not in fact emerge until the next approximation is considered.

The terms of order σ yield

$$\begin{aligned}(\theta_t + c\theta_x^{(0)})\,\eta_\theta^{(1)} + \tfrac{3}{2}ch^{-1}\theta_x^{(0)}(\eta^{(0)}\eta^{(1)})_\theta + \tfrac{1}{6}h^2c\theta_x^{(0)3}\eta_{\theta\theta\theta}^{(1)} \\ = -(\tfrac{1}{2}c_X\eta^{(0)} + c\eta_X^{(0)})\,\epsilon^2 - \theta_x^{(1)}(c\eta_\theta^{(0)} - \tfrac{3}{2}ch^{-1}\eta^{(0)}\eta_\theta^{(0)} - \tfrac{1}{2}h^2c\theta_x^{(0)2}\eta_{\theta\theta\theta}^{(0)}).\end{aligned} \quad (8)$$

In $(5a, b)$ we have $\eta^{(0)} = O(H) = O(\epsilon^3)$ and $\theta_x^{(0)} = O(\theta_x) = O(1)$. Thus if $(5a, b)$ are asymptotic expansions in the small parameter σ, we must have $\eta^{(1)} \lesssim \eta^{(0)}$ and $\theta_x^{(1)} \lesssim \theta_x^{(0)}$. Inspection of (8), however, shows that $\theta_x^{(1)} \sim \theta_x^{(0)}$ cannot be accepted, because then the term $\theta_x^{(1)}c\eta_\theta^{(0)}$ is $O(\epsilon^3)$, which is inconsistent with the fact that all other terms in (8) are $O(\epsilon^5)$. So we must require that $\theta_x^{(1)} = O(\epsilon^2)$ at most; consequently the last two terms in (8) can be omitted.

Having established the solution for $\eta^{(0)}$, (8) can be solved for $\eta^{(1)}$, which is readily done since (8) can be integrated once directly and the resulting second-order equation does not contain θ explicitly. The complete solution to (8) is

$$\eta^{(1)} = \eta_\theta^{(0)}\left[C(X) + \int \frac{\mathscr{F}(\theta, X)}{\eta_\theta^{(0)2}}\,d\theta - \frac{3\theta_x^{(1)}}{2h^2\theta_x^{(0)3}}\int \frac{\eta^{(0)2}}{\eta_\theta^{(0)2}}\,d\theta\right], \quad (9a)$$

where

$$\mathscr{F}(\theta, X) = \int \eta_\theta^{(0)}(F(\theta, X) + F_1(X))\,d\theta + F_2(X), \quad (9b)$$

$$F(\theta, X) = \alpha\epsilon^2 \int (c_X\eta^{(0)} + 2c\eta_X^{(0)})\,d\theta, \quad (9c)$$

$$\alpha = -3/(h^2\theta_t\theta_x^{(0)2}), \quad (9d)$$

C, F_1 and F_2 being arbitrary functions.

The value of $\theta_x^{(1)}$ and the arbitrary functions $F_1(X)$ and $F_2(X)$ are determined by requiring that both $\eta^{(1)}$ and $\eta_\theta^{(1)}$ should be continuous and bounded at the points where $\eta_\theta^{(0)} = 0$, i.e. at the crest and trough of the first-order solution. It turns out that this is the case provided that

$$\theta_x^{(1)} = F_1(X) = F_2(X) = 0.$$

Now the only arbitrary function left is $C(X)$. First it is realized that $\eta_\theta^{(0)}$ is a solution to the homogeneous (i.e. horizontal-bottom) equation for $\eta^{(1)}$, so the term $C(X)\,\eta_\theta^{(0)}$ can be non-zero only if it is required to suppress secular terms in the solution to the $O(\sigma^2)$ equation (Cole 1968, p. 82; Reiss 1971). Further, since we are dealing with a regular perturbation problem in h_x, a non-zero $C(X)$ must vanish with h_x.

It may be shown that the equation for $\eta^{(2)}$ will have the same differential operator as that for $\eta^{(1)}$, and thus a term such as $C(X)\,\eta_\theta^{(0)}$ will be a resonant forcing term in the equation for $\eta^{(2)}$. The other possible resonant term has the form $\eta_\theta^{(0)}\int \theta_\theta^{(0)-2}\,d\theta$ and it turns out that neither of these resonant terms appears in the inhomogeneous part of the equation for $\eta^{(2)}$. Therefore, assuming the proper initial conditions such that $\eta^{(1)} \to 0$ as $h_x \to 0$ and $\theta = 0$ at the crest of the wave, we may put $C(X) = 0$ in (9). We then get

$$\eta^{(1)} = \eta_\theta^{(0)}\int_0^\theta \frac{\mathscr{F}(\tilde{\theta}, X)}{\eta_\theta^{(0)2}}\,d\tilde{\theta} + O(\epsilon^5), \quad (10)$$

where
$$\mathscr{F}(\theta, X) = \frac{3\epsilon^2}{h^2\theta_t \theta_x^{(0)2}} \int_0^\theta \eta_\theta^{(0)} \int_0^\theta (c_X \eta^{(0)}(\hat{\theta}, X) + 2c\eta_X^{(0)}(\hat{\theta}, X)\, d\hat{\theta}\, d\tilde{\theta} \tag{11}$$

and η is given by (5a).

Though (10) and (11) give the final solution it still remains to discover what constraint should be placed on the X dependence of c, c_X, $\eta^{(0)}$ and $\eta_X^{(0)}$ in order to ensure periodicity of the solution. Since $\mathscr{F} = \eta_\theta^{(0)}\eta_\theta^{(1)} - \eta^{(1)}\eta_{\theta\theta}^{(0)}$ from (10), periodicity of $\eta^{(1)}$ implies periodicity of $\mathscr{F}$, i.e.

$$\mathscr{F}(0, X) = \mathscr{F}(1, X) = 0. \tag{12}$$

Integrating (11) by parts shows that this condition may be written as

$$\int_0^1 \eta_{\tilde{\theta}}^{(0)} \int_0^{\tilde{\theta}} (c_X \eta^{(0)}(\hat{\theta}, X) + 2c\eta_X^{(0)}(\hat{\theta}, X))\, d\hat{\theta}\, d\tilde{\theta}$$
$$= \left[\eta^{(0)} \int (c_X \eta^{(0)} + 2c\eta_X^{(0)})\, d\theta\right]_0^1 - \int_0^1 (c\eta^{(0)2})_X\, d\theta = 0, \tag{13}$$

and since the first term can be shown to be zero this corresponds to requiring

$$\int_0^1 c\eta^{(0)2}\, d\theta \equiv \overline{c\eta^{(0)2}} = \text{constant}. \tag{14}$$

Here $\overline{c\eta^{(0)2}}$ corresponds to the mean energy flux F (Ostrovskiy & Pelinovskiy 1970), so that (14) formally proves that the well-known principle of constancy of energy flux must apply to the first-order solution. This constraint and the constancy of the wave period together supply the extra condition required to determine the variation of H and m in (7).

3. Numerical results for $\eta^{(1)}$

The evaluation of (10) and (11) is rather complicated because it involves the derivatives with respect to X of the varying coefficients H and β and of the cn function in (7a), determined under the constraints (14) and (7f). It turns out to be convenient to express the variations with respect to X through the slowly varying quantity $U(X) \equiv HL^2/h^3$. To do this we eliminate other variables from (7) and (14) using the identity $L = c_1 T$ and obtain a transcendental 'shoaling equation' for U. Though we found that the local phase velocity c_1 is given by (7), consistency requires that in the calculation of $\eta_X^{(0)}$ we use c for the local value of c_1. Then the 'shoaling equation' can be written as

$$U = \tfrac{1}{4}(2\pi)^{-\frac{3}{2}}(H_0/L_0)\,(\epsilon h/T^2)^{\frac{9}{4}} B^{-\frac{1}{2}}, \tag{15a}$$

where
$$B = \frac{1}{3m^2}\left[m - 1 + 2(2-m)\frac{E}{K} - 3\left(\frac{E}{K}\right)^2\right]. \tag{15b}$$

H_0/L_0 is the deep-water wave steepness (which is essentially a measure of the mean energy flux F) and B is defined by $\overline{\eta^{(0)2}} = H^2 B$.

The constraint (15) is invoked by transforming the X derivatives in (11) into derivatives with respect to U, which yields

$$\sigma\epsilon^2 \frac{\partial}{\partial X} = h_x \frac{\partial}{\partial h}\bigg|_{\theta=\text{const}} = h_x \frac{dU}{dh}\frac{\partial}{\partial U}\bigg|_{\theta=\text{const}}, \tag{16}$$

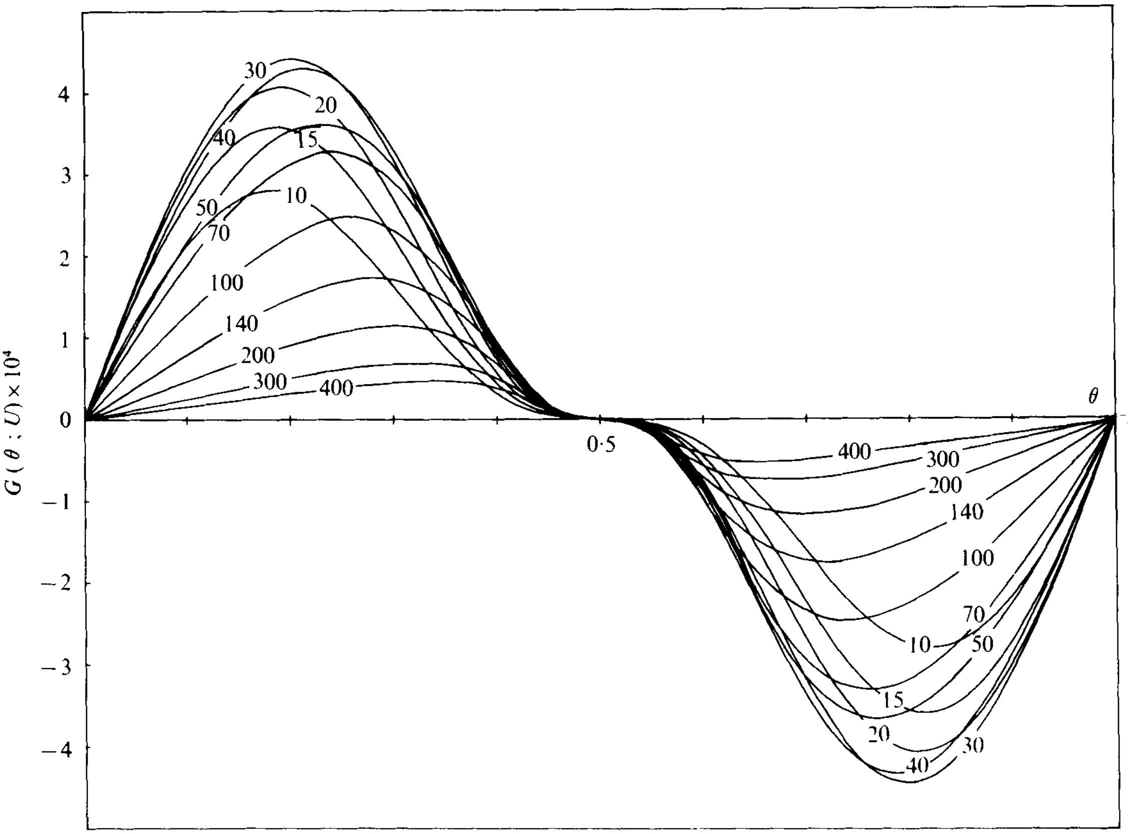

FIGURE 2. The function $G(\theta;\,U)$ in (21) as a function of the phase angle θ. Each curve is marked with the value of $U = HL^2/h^3$.

where dU/dh is determined by implicit differentiation of (15). The rest of the computation consists of calculating the derivatives with respect to h or U of c, $\eta^{(0)}$, m and B. This is not discussed in detail here. However, it may be noted that the quantity $\xi = \partial(\operatorname{cn}(2K\theta, m))/\partial m$ appearing in $\eta_X^{(0)}$ must be evaluated for θ = constant. Byrd & Friedmann (1971, equation 710.52) give $\partial(\operatorname{cn}(u,m))/\partial m$ for $u = 2K\theta$ = constant. From this ξ may be determined by realizing that

$$\xi = \left(\frac{\partial(\operatorname{cn} u)}{\partial m}\right)_{\theta=\text{const}} = \left(\frac{\partial(\operatorname{cn} u)}{\partial m}\right)_{u=\text{const}} - \operatorname{sn} u \operatorname{dn} u \frac{u}{2mm_1}\left(\frac{E}{K}-m\right). \quad (17)$$

For presentation of the numerical results it turns out to be convenient to write $\sigma\eta^{(1)}$ as

$$\sigma\eta^{(1)} = 3(L/h)^2 h_x T\, c f(\theta;\, U, H/h), \quad (18)$$

where $L = \theta_x^{(0)-1}$ is the wavelength, $T = \theta_t^{-1}$ is the wave period and where the arguments U and H/h of f have been included explicitly. f is defined by

$$f(\theta;\,U,H/h) = \frac{\eta_\theta^{(0)}}{c}\int_0^\theta \eta_{\check\theta}^{(0)-2}\int_0^{\check\theta}\eta_{\tilde\theta}^{(0)}\int_0^{\tilde\theta}[2(c\eta^{(0)})_h - c_h\eta^{(0)}]\,d\hat\theta\, d\tilde\theta\, d\check\theta. \quad (19)$$

It may further be shown that f can be written as

$$f(\theta; U, H/h) = (H/h)\, G(\theta; U), \tag{20}$$

whereupon (18) can be written as

$$\sigma\eta^{(1)}/H = 3(L/h)^2 h_x T(g/h)^{\frac{1}{2}} G(\theta; U). \tag{21}$$

Note that, if H/h and one of the variables L/h, $T(g/h)^{\frac{1}{2}}$ and U are given, the remaining two may be determined, so in addition to h_x the problem has only two parameters, corresponding to the two parameters required to specify the unperturbed wave.

Results for the function G are shown in figure 2, plotted against θ. Since X (i.e. x) is constant, variation of θ corresponds to variation of the time t; values of U are marked on the curves.

It may be shown that in the neighbourhood of the wave crest $\sigma\eta^{(1)}$ varies as θ^3, which implies that no tilt in the wave profile occurs there. On the other hand, in the wave trough $\sigma\eta^{(1)}$ varies approximately linearly with θ, which implies a sloping surface profile in the trough with the deepest point just before the next wave crest. Thus the resulting wave is actually more skewed in the wave trough than around the wave crest. This fact will be confirmed by direct comparison with experimental results.

4. Comparison with experiments and discussion

The experiments used for comparison were done in a wave flume 60 cm wide. A plane beach had been installed whose slope was $\frac{1}{35}$, which is a representative value for many natural beaches. The waves were generated by a piston-type wave generator whose motion ζ was given by

$$\zeta = a_1 \cos \omega t + a_2 \cos(2\omega t + \beta_2), \tag{22}$$

where the constants a_2 and β_2 were chosen such that the free second-harmonic waves in the flume were reduced to a minimum. In this way very regular waves were obtained (Buhr Hansen & Svendsen 1974). The surface variations were measured by a resistance-type wave transducer (two silver wires 0·17 mm in diameter and 5 mm apart), the signal from which was scanned 400 times per second by a PDP8 minicomputer working on-line during the experiment.

Figure 3 shows measured and theoretical profiles (both cnoidal and second-order Stokes) for one of the wave trains used in the following comparison. It should be noticed, however, that the property of real importance here is not the agreement with some theory but the constant form of the waves indicated in figure 3 by the absence of free secondary wavelets.

4.1. *Wave-height variation*

The variation of the wave height was discussed by Svendsen & Buhr Hansen (1977) and a brief account of their results may be relevant here. They measured the vertical distance between the highest and the lowest points of the surface profile using a transducer on a carriage moving slowly along the flume. The measured quantities were compared with the theoretical results for H. Figure 4 shows the variation on the

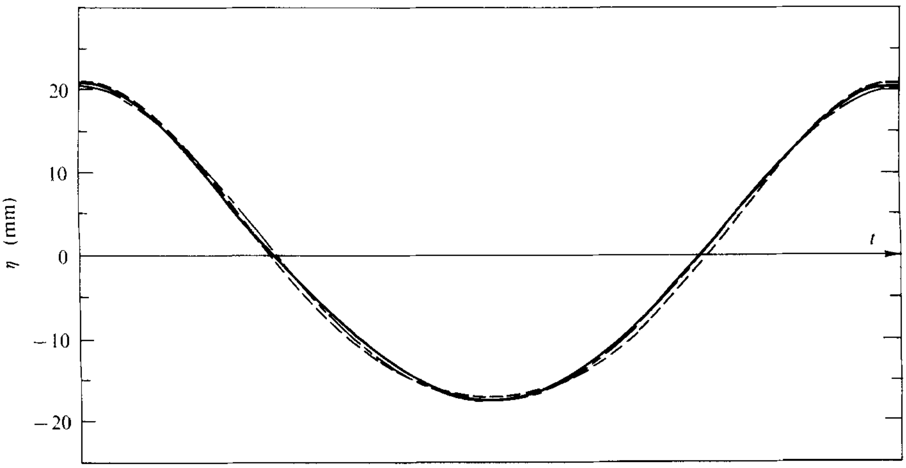

FIGURE 3. Measured and theoretical wave profiles for a constant depth. ——, measured; ——, Stokes second-order theory; ———, first-order cnoidal theory. $H_0/L_0 = 0{\cdot}0099$.

slope up to the point of breaking. We see that for the curves corresponding to (15) (solid lines) the agreement is quite good although H/h at breaking is about 0·9, which is far beyond the value for which a cnoidal theory should work. This does not mean, however, that the 'small' terms omitted are insignificant, as is clearly seen from the other curves in figure 4, which show the variation of H resulting from two (inconsistent) modifications to the theory. The first (dot–dash lines) is based on a local phase velocity given by (7) in connexion with (14). This version (suggested by Ostrovskiy & Pelinovskiy 1970) predicts values of the wave height which are somewhat too high (by 20–30 % at the point of breaking). Also this version does not satisfy (12), which is crucial for the numerical calculation of $\eta^{(1)}$. In order to satisfy (12) with a local phase velocity equal to c_1 as given by (7), we have to use c_1 instead of c in (14). As the dashed lines in figure 4 show, the agreement of this second version, which was originally investigated by Svendsen & Brink-Kjær (1972) is surprisingly good.

Owing to the deformation of the wave profile [corresponding to the $\sigma\eta^{(1)}$ term in (5)] the measured vertical distance between the highest and lowest points of the profile is slightly larger than H in (7). This effect is not included in the theoretical curves in figure 4, and would increase the predicted values by up to 5 % at the point of breaking.

It should be added that the theoretical curves in figure 4 have been corrected for the energy losses in the flume, assuming laminar boundary layers and the particle velocities according to linear wave theory.

4.2. *Wave profiles*

Wave profiles were obtained at a number of locations on the slope and three series of such profiles are shown in figures 5, 6 and 7. The measured profiles are compared with theoretical wave profiles determined from (5), (7), (10) and (11). For the evaluation of these profiles we used the measured water depth (corrected for wave set-down) at the point of the flume where the profile was measured, the wave frequency and the bottom slope. The last parameter required to determine the wave profile λ is H in (7). In figures 5, 6 and 7 the solid curves show numerical results based on a value of H

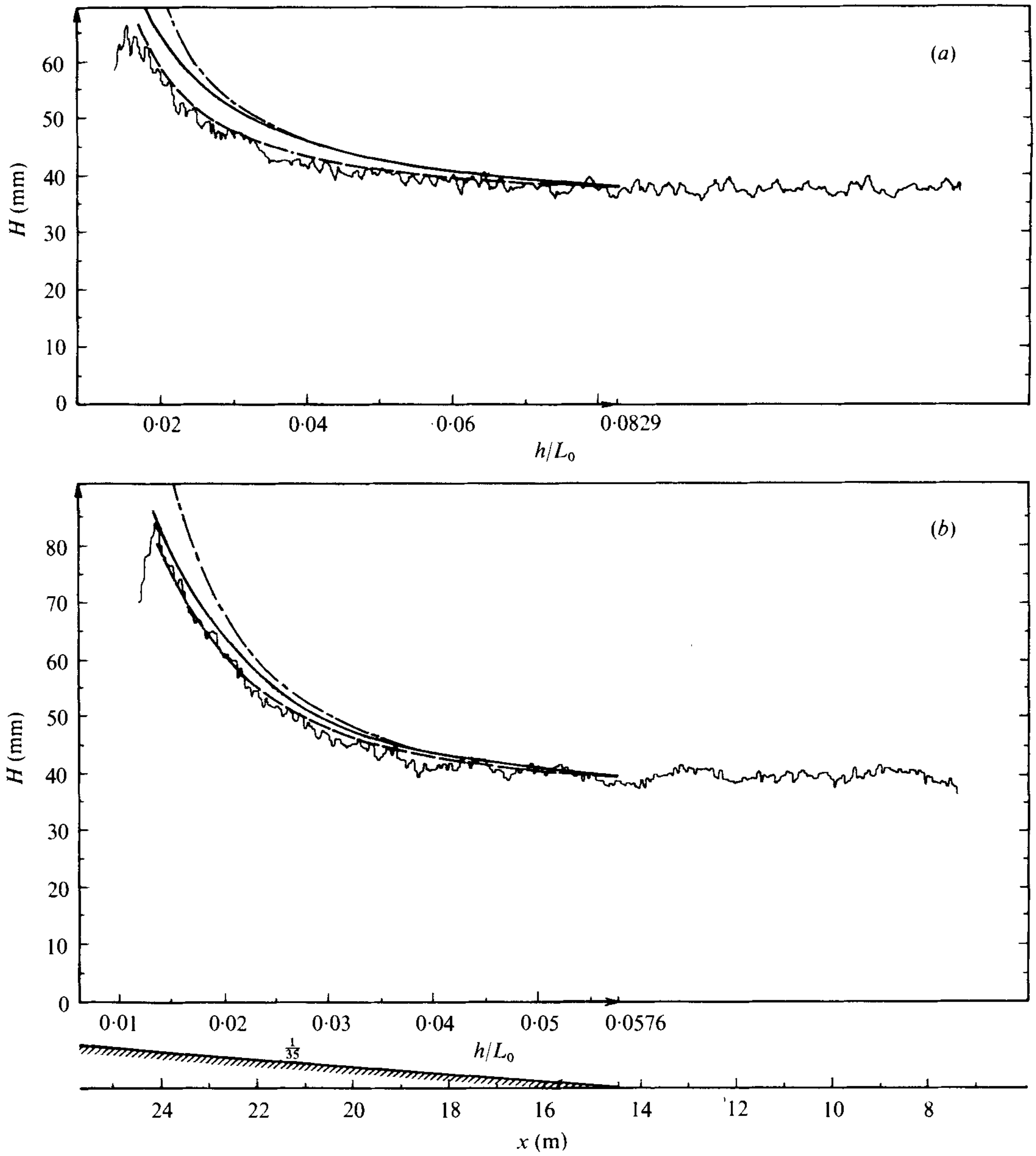

FIGURE 4. The wave-height variation over a slope of $\frac{1}{35}$. Irregular curve, measured values; ——, theoretical values based on (15); — — —, theoretical values based on (7) and (14); — —, theoretical values based on (7), and on (14) with c_1 instead of c. (a) $H_0/L_0 = 0{\cdot}099$. (b) $H_0/L_0 = 0{\cdot}0039$. (From Svendsen & Buhr Hansen 1977.)

which for each profile was determined such that the vertical distance between the wave crest and the deepest point of the wave trough, i.e. the 'wave height', was the same in the measured and the calculated profiles. On the other hand, in the deformed wave profile H can also be determined as $\eta(0) - \eta(\frac{1}{2})$ and this value of H has been used in the calculations shown by a dotted line in figure 7 (the corresponding curves are

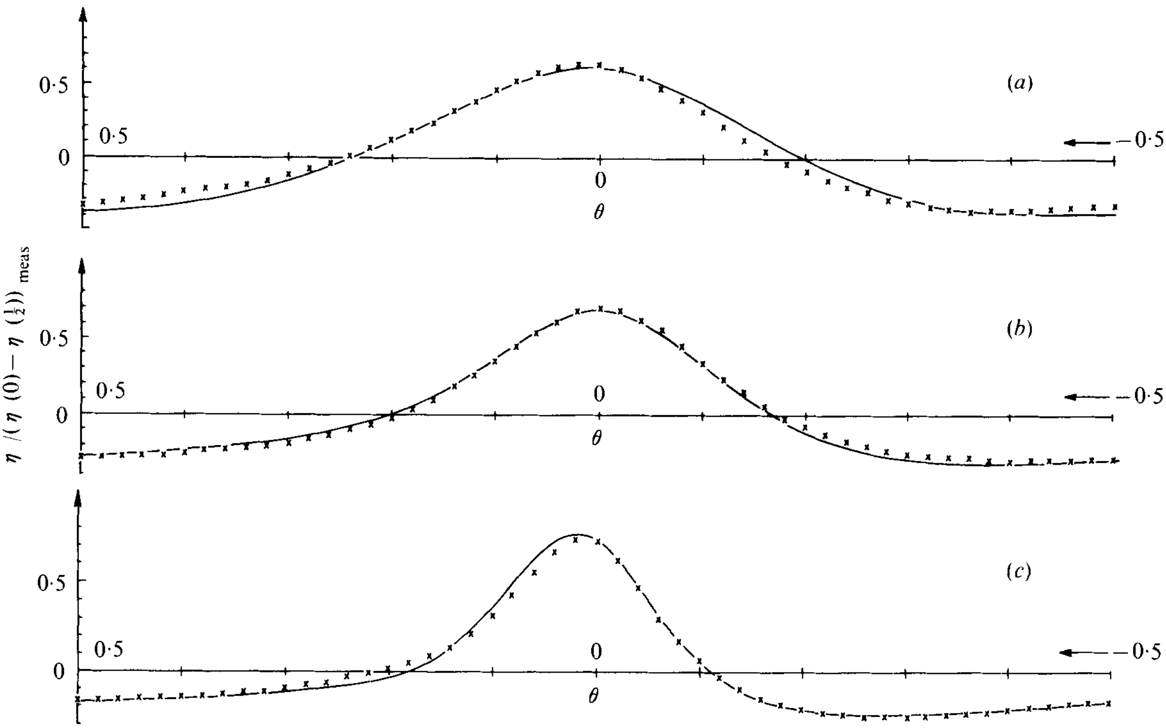

FIGURE 5. Comparison of the measured (crosses) and predicted (curves) wave profiles at various depths for an experiment with $\theta_t = 0{\cdot}6$. Bottom slope $= \frac{1}{35}$, $H_0/L_0 = 0{\cdot}0165$. (*a*) $H/h = 0{\cdot}27$, $L/h = 9{\cdot}62$, $U = 25{\cdot}0$. (*b*) $H/h = 0{\cdot}389$, $L/h = 12{\cdot}4$, $U = 59{\cdot}8$. (*c*) $H/h = 0{\cdot}585$, $L/h = 15{\cdot}3$, $U = 137$.

not included in figures 5 and 6 as they can hardly be distinguished from the solid curves).

For both the measured and the calculated profiles $\eta = 0$ corresponds to the mean water level, i.e.

$$\int_0^T \eta \, dt = 0,$$

and the profiles have been aligned in θ by setting $\theta = 0$ at the wave crests of both profiles.

The parameters H/h and L/h (and U, which is derived from them) are also given for each profile. It may be seen that the wavelength is generally more than 10 times the water depth, 9·62 being the lowest value (figure 5), which means that in all cases we are above the deep-water limit for cnoidal waves. Below this limit, which corresponds to a value of L/h close to 5, no cnoidal solution exists (Svendsen & Buhr Hansen 1977).

Figure 5 shows three profiles for which H/h is between 0·27 and 0·585. The agreement is convincing even though H/h is appreciable.

The wave series shown in figures 6 and 7 are included to illustrate the limitations of the theory. The last profiles in each of these figures were taken at the visually determined breaking point. For the wave train shown in figure 6 the general impression

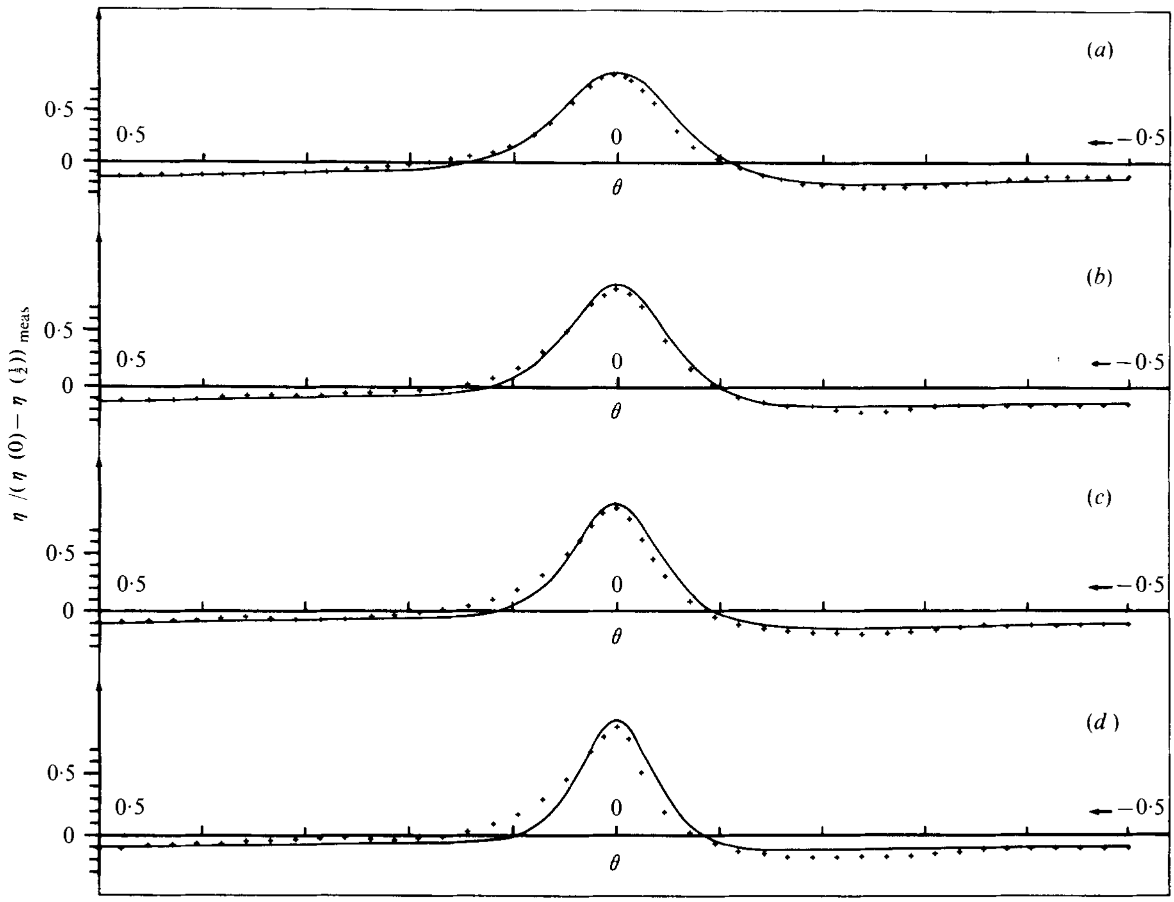

FIGURE 6. Comparison of measured (crosses) and predicted (curves) wave profiles at various depths for an experiment with $\theta_t = 0{\cdot}6$. Bottom slope $= \frac{1}{35}$, $H_0/L_0 = 0{\cdot}0099$. (*a*) $H/h = 0{\cdot}579$, $L/h = 20{\cdot}1$, $U = 234$. (*b*) $H/h = 0{\cdot}673$, $L/h = 21{\cdot}5$, $U = 311$. (*c*) $H/h = 0{\cdot}736$, $L/h = 22{\cdot}9$, $U = 386$. (*d*) $H/h = 0{\cdot}871$, $L/h = 25{\cdot}1$, $U = 549$.

is that the agreement is good up to H/h values of 0·7 or 0·8 (i.e. figure 6*b* or perhaps figure 6*c*), whereafter the deviations increase rapidly, though the results do not become directly misleading even at the breaking point. The deviation is particularly large where $\sigma\eta^{()}$ is large, indicating that close to breaking the effect of the sloping bottom is underestimated by the theory.

In contrast to this the profiles in figure 7 show appreciable differences between theoretical and measured values, even for $H/h = 0{\cdot}619$, although the calculation with H based on $\eta(0)-\eta(\frac{1}{2})$ is slightly better. The reason for this is probably the much larger values of L/h for these waves, which indicate that the bottom slope of $h_x = \frac{1}{35}$ is too steep for these waves to satisfy the requirement $h_x \ll \epsilon^3$ or $\sigma \ll 1$ (see §1). Recalling that in dimensional terms $\epsilon = h_0'/\lambda' \sim h'/L'$, we see that this result is quite in accordance with the well-known fact that a given slope h_x may appear 'gentle' to a train of waves with relatively small L'/h' but 'steep' to another train of much longer waves. The conjecture that h_x is not $\ll \epsilon^3$ in the experiment shown in figure 7

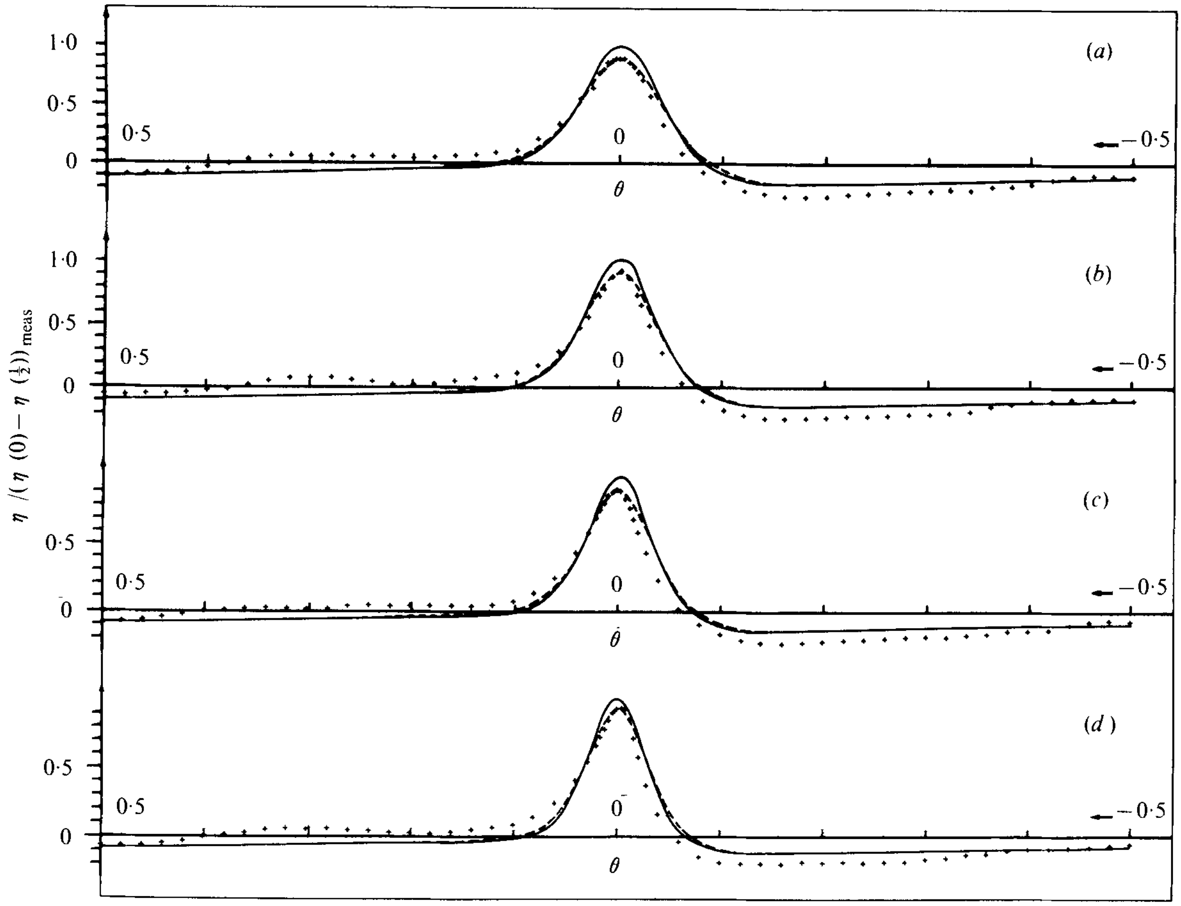

FIGURE 7. Comparison of measured (crosses) and predicted (curves) wave profiles at various depths for an experiment with $\theta_t = 0{\cdot}4$. Bottom slope $= \frac{1}{35}$, $H_0/L_0 = 0{\cdot}0039$. (*a*) $H/h = 0{\cdot}619$, $L/h = 28{\cdot}9$, $U = 517$. (*b*) $H/h = 0{\cdot}672$, $L/h = 30{\cdot}5$, $U = 625$. (*c*) $H/h = 0{\cdot}732$, $L/h = 31{\cdot}5$, $U = 726$. (*d*) $H/h = 0{\cdot}797$, $L/h = 33{\cdot}3$, $U = 884$. ——, total wave height the same as that measured; - - - - -, $H = \eta(0) - \eta(\frac{1}{2})$.

is further supported by the somewhat different natures of the measured profiles, which have a small but unmistakable secondary crest behind the main crest.

A peculiar aspect of the theory is the behaviour of (21) for relatively large values of U. From figure 2 and from (21) it can be inferred that the maximum value in $0 < \theta < 1$ of the relative deformation $\sigma\eta^{(1)}/H$ occurs for $U \sim 50$. For a particular wave train increasing U means decreasing h, so that the theory actually predicts a (slowly) decreasing relative deformation as the wave approaches breaking for $U \gtrsim 50$.

The measurements show a tendency which is similar though not quite so pronounced. For the measured profiles the maximum value of the deformation can be estimated as $\max \frac{1}{2}\{\eta(\theta) - \eta(-\theta)\}$ ($\theta = 0$ at the wave crest), and for the experiment in figure 6 this quantity is almost constant at $0{\cdot}10$ ($\pm 10\,\%$) for U ranging between 25 and 400. Above $U \sim 400$ the relative deformation increases rather rapidly and the wave breaks at $U \sim 550$.

5. Conclusion

The paper treats the problem of periodic long waves in water of slowly varying depth. A solution is determined for the wave profiles which to the first order is a slowly varying cnoidal wave. The second approximation, given by (10), represents the deformation of the wave profile due to the sloping bottom. This is assumed small in comparison with the wave itself. As a by-product the wave-height variation due to the changing water depth can be also determined.

The solution is shown to be valid for a bottom slope in the interval $\epsilon^5 \ll h_x \ll \epsilon^3$. In that case the wave deformation at any point of the slope will be determined by the local water depth and bottom slope. Thus even though the waves are deformed by the sloping bottom they do not disintegrate into solitons. This situation represented by the theoretical model will in nature eventually lead to wave breaking on the beach.

Numerical results are presented in figure 2 and from these the solution for the wave profile can be evaluated using (5), (7) and (21). Comparison with experiments on regular waves over a plane of slope $h_x = \frac{1}{35}$ shows fair agreement both for the wave-height variation (figure 4) and for the wave profiles (figures 5–7) though the discrepancies naturally increase for very large values of the ratio H/h of wave height to water depth. As a general result we also notice that the sloping trough predicted by the theory as the most significant effect of the sloping bottom is even more pronounced in the experimental results.

Finally, the practical limit of validity of the assumption for the bottom slope is discussed. The parameter of primary importance for the relative deformation is shown to be σ (see (2), § 1), which means that the ratio ϵ^{-1} of wavelength to water depth is just as important as the bottom slope h_x. It is suggested that owing to the large values of ϵ^{-1} the waves in figure 7 do not in fact satisfy the criterion $\sigma \ll 1$.

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